TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 295, Number 1, May 1986

DEFINING EQUATIONS FOR REAL ANALYTIC REAL HYPERSURFACES IN $\mathbf{C}^n$

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ABSTRACT. A defining function for a real analytic real hypersurface can be uniquely written as $2\operatorname{Re}(H) + E$, where H is holomorphic and E contains no pure terms. We study how H and E change when we perform a local biholomorphic change of coordinates, or multiply by a unit. One of the main results is necesary and sufficient conditions on the first nonvanishing homogeneous part of E (expanded in terms of H) beyond E_{00} that serve as obstructions to writing a defining equation as $2\operatorname{Re}(h) + e$, where e is independent of h. We also find necessary pluriharmonic obstructions to doing this, which arise from the easier case of attempting to straighten the hypersurface.

I. Introduction. This paper is the first of several in which we consider the defining equation for a real analytic real hypersurface in $\mathbf{C}^n$. The ultimate goal of this work would be to find normal forms for such equations, but this is probably intractable in general.

Tanaka [**13**] and Chern-Moser [**7**] have found such normal forms for strictly pseudoconvex hypersurfaces; there is considerable interest in the Chern-Moser invariants because of their relationship with the analysis on the domain bounded by the hypersurface (see the survey article [**2**]). On the other hand, the analysis on weakly pseudo-convex domains is becoming well understood. For example, suppose that Ω is a smoothly bounded pseudoconvex domain, and the order of contact of complex varieties with the boundary hypersurface is bounded [**8**]. Catlin [**5, 6**] has shown that this condition is necessary and sufficient for certain subelliptic estimates of Kohn. These estimates imply condition R of Bell [**3**], and hence that biholomorphic mappings extend smoothly to the boundaries. Therefore the connection between (boundary) invariants of hypersurfaces and complex function theory persists. It is natural, therefore, to seek "partial" normal forms for the defining equations, and to determine from them as much information as possible.

The present work considers arbitrary real analytic real hypersurfaces, at a single point. We study how the defining function changes when we perform a local biholomorphic change of coordinates, or when we multiply by a unit, obtaining a new defining function. Assume that p is the origin, and that we write our defining function as $2\operatorname{Re}(H)+E$, where H is holomorphic, E is real analytic, and E contains no pure terms in its Taylor expansion. This determines H and E uniquely. After choosing a coordinate system containing H as a member, we can write

$$(*) \qquad E = \sum_{0,0}^{\infty} E_{ik} H^i \overline{H}^k, \quad \text{where } E_{ik} \text{ is independent of } H.$$

Received by the editors February 27, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 32F25; Secondary 32B05.
[1]Partially supported by the NSF.

In §III we compute the index of flatness, namely the largest integer m for which there is a defining function for which we can write $(*)$, where $E_{ik} = 0$ for $i+k < m$. In §IV we consider the more interesting case where $E_{00} \neq 0$. There we compute the index of Tanaka regularity, namely the largest integer m for which there is a defining function for which we have $E_{ik} = 0$ for $0 < i+k < m$. Our main result is

THEOREM IV.8. *Suppose that M is a real analytic hypersurface, defined by $2\operatorname{Re}(H) + E$ at p, $m \geq 2$, and $E_{ik} = 0$ for $0 < i+k < m$, but $E_{00} \neq 0$. Write E_m for the homogeneous part of order m. Then there is a new defining function, $2\operatorname{Re}(h) + e$, and new coordinates, for which $e_k = 0$ for $1 \leq k \leq m$, if and only if there are t real analytic, and a holomorphic so that (8.1) holds (t and a are independent of H).*

$$E_m \equiv t(M + \overline{H})^m + 2\operatorname{Re}(aH^{m-1}\overline{H}) \mod(E_{00}). \tag{8.1}$$

This congruence is a strong condition on E_m, because the first term lies in the ideal $(H+\overline{H})^m$ rather than simply $(H+\overline{H})$. In the preliminary section we see that the ideals (H) and (E_{00}) are invariants. We offer several corollaries. One is that, if it is possible to write a defining function in the T-regular form $2\operatorname{Re}(H) + E_{00}$, then there is a unique way of doing so. We use this result in conjunction with the expression $E_{00} = \|F\|^2 - \|G\|^2$ to determine when our defining function has the form $2\operatorname{Re}(z_n) + \sum^N |f_j(z')|^2$. Also, (8.1) gives us certain "pluriharmonic" obstructions to T-regularizing.

There are numerous questions left untouched here. We offer no proofs of convergence, and we do not go on once we have T-regularized the equation. We have obtained partial results on this last point, which we hope to complete in a future paper. The methods here can also be used in higher codimension. There are some known results in special cases [**11, 12**]. Finally, the process of T-regularizing, in the higher codimension case, is related also to work of Baouendi-Rothschild on CR functions [**1**].

II. Preliminaries. Let M be a real analytic real hypersurface of $\mathbf{C}^n$. We want to extract geometric information about M from a defining equation, but we hope to express this information in a fashion that is independent of the choice of defining function or local holomorphic coordinates. We require some algebraic preliminaries about rings and ideals of germs of functions at a point p in M.

1. NOTATIONS. $\mathcal{O}_p$ denotes the local ring of germs of holomorphic functions at p; its maximal ideal is $\mathcal{M}_p$. $\mathcal{A}_p$ denotes the local ring of germs of real analytic functions at p; its maximal ideal is generated by $\mathcal{M}_p$ and $\overline{\mathcal{M}}_p$. Here the bar denotes complex conjugation. We also need the ideal $|\mathcal{M}_p|^2$, which consists of finite sums of products of elements in $\mathcal{M}_p$ and $\overline{\mathcal{M}}_p$. An alternate description is those germs that vanish at p, and contain no pure terms in their Taylor expansions there. We remark that such germs are not necessarily real valued.

The proofs of the following lemmas are elementary, and hence omitted.

2. LEMMA. *Let $r \in \mathcal{A}_p$. Then there is a unique decomposition of r:*

$$r = r(p) + \alpha + \overline{\beta} + e, \tag{2.1}$$

where $\alpha, \beta \in \mathcal{M}_p$ and $e \in |\mathcal{M}_p|^2$. Suppose also that r is real valued. Then $\alpha = \beta$.

3. LEMMA. *Let u be a real valued unit in $\mathcal{A}_p$. Then there is a unique factorization of u:*

$$u = u(p)(1 + 2\,\mathrm{Re}(a))(1 + B), \tag{3.1}$$

where $a \in \mathcal{M}_p$, $B \in |\mathcal{M}_p|^2$, and B is real valued.

In case $r(p)$ vanishes, we write $\pi_{10}(r) = \alpha$ and $\pi_{01}(r) = \beta$. If also r is real valued, we write the decomposition (2.1), $r = 2\,\mathrm{Re}(h) + e$, also as $\langle h, e\rangle$. Suppose that r is a defining function for M near p; that is, r is a real valued generator for the principal ideal of functions that vanish on M. Then since M is a hypersurface, $dr(p)$ does not vanish. This implies that $dh(p)$ does not vanish either.

Suppose we multiply r by a real valued unit. We obtain a new defining function, say $\langle H, E\rangle$. They are related by the following often used proposition:

4. PROPOSITION. *$\langle h, e\rangle$ and $\langle H, E\rangle$ define the same hypersurface if and only if there exist $a \in \mathcal{M}_p$, $B \in |\mathcal{M}_p|^2$, and a nonvanishing real number c so that*

$$H = h(1 + a)c, \tag{4.1}$$

$$E = e(1 + 2\,\mathrm{Re}(a))(1 + B)c + B \cdot 2\,\mathrm{Re}(H) + (1 + B)|1 + a|^{-2} 2\,\mathrm{Re}(H\overline{\alpha}), \tag{4.2}$$

where $\alpha = a + a^2$.

PROOF. By Lemma 3, we can decompose the unit u as

$$u = c(1 + 2\,\mathrm{Re}(a))(1 + B). \tag{4.3}$$

We multiply (4.3) by $\langle h, e\rangle$, and collect pure terms. This forces (4.1). If we substitute for h using (4.1), and simplify the terms in $|\mathcal{M}_p|^2$, (4.2) results.

5. REMARKS. Henceforth we assume without loss of generality that c equals one. Also, we assume that all terms in (4.2) are expressed in terms of H (rather than h).

6. COROLLARY. *Let M be a real analytic hypersurface in $\mathbf{C}^n$. For each p in M, there is a complex analytic hypersurface $W(p)$ defined by the vanishing of $\pi_{10} r$ for any defining function r. $W(p)$ is independent of r, nonsingular at p, and depends real analytically on p.*

PROOF. By (4.1), $\pi_{10}(ur)$ is a unit in $\mathcal{O}_p$ times $\pi_{10}(r)$. Hence the ideal $(\pi_{10}(r))$ in $\mathcal{O}_p$ is an invariant of M at p. Its variety is also an invariant, and is nonsingular because $dh(p)$ does not vanish. Finally, the assignment $p \mapsto \pi_{10}(r)$ at p is defined by Taylor coefficients of r, and hence depends (real) analytically on p.

Suppose now that z_H denotes a local holomorphic coordinate system near p, for which H is one coordinate in this system. We can expand E in terms of H; of course the expansion depends on z_H. We use the following notation when z_H is fixed:

$$E = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} E_{ik} H^i \overline{H}^k; \quad E_{ik} \text{ are real analytic,} \quad \frac{\partial E_{ik}}{\partial H} = 0, \tag{7.1}$$

$$E_m = \sum_{i+k=m} E_{ik} H^i \overline{H}^k, \tag{7.2}$$

$$j_k(E, z_H) = j_k(E) = \sum_{m=0}^{k} E_m; \quad \text{also } j_{-1}(E) = 0 \text{ by definition,} \tag{7.3}$$

$$\text{Alt}(E_m) = (\sqrt{-1})^m \sum_{k=0}^{m} E_{k(m-k)}(-1)^k. \tag{7.4}$$

Note that the statement "$j_k(E)$ vanishes" does not depend upon the choice of coordinates z_H. However, a new defining function, and corresponding new coordinates z_h, could make $j_s(e, z_h)$ vanish for some larger s. These considerations lead to the following:

8. DEFINITION. Let M be a real analytic hypersurface containing p. The index of flatness, $F(M,p)$, is the largest integer m for which there is a defining function $\langle H, E\rangle$ for which $j_{m-1}(E, z_H)$ vanishes. If no such integer exists, we say $F(M,p)$ equals infinity. We also use the phrase that $\langle H, E\rangle$ exhibits M as flat to order m at p if $j_{m-1}(E, z_H)$ vanishes.

Note that $F(M,p)$ is an invariant of M at p. If we are given a defining function, however, that exhibits M as flat to order m at p, we need to determine whether or not $F(M,p)$ actually equals m. This is the purpose of §III. Before getting to these results, we need one more preliminary and we make several remarks.

9. LEMMA. *Let $G \in \mathcal{A}_p$, and suppose that $G = G_m$. (There is a fixed holomorphic coordinate system z_H.) Then*

$$\text{Alt}(G_m) = 0 \Leftrightarrow G \textit{ is in the ideal } (H + \overline{H}). \tag{9.1}$$

PROOF. We perform long division by G_m of $H + \overline{H}$. The remainder is a power of $\sqrt{-1}$ times $\text{Alt}(G_m)$. Both directions of the implication follow.

10. REMARK. The index of flatness does not measure the order of vanishing of E. For example, if $r = 2\,\text{Re}(z_2) + |z_1|^{2k}$ and p is the origin, then $F(M,p)$ equals zero for every positive integer k.

11. REMARK. Since E is real valued, $E_{ik} = \overline{E}_{ki}$. Thus E_{ik} can be considered as a Hermitian form on the Hilbert space with basis $\{H^k\}$. The author exploits this fact when investigating the order of contact of complex analytic varieties [**9**]. Here we need additional considerations.

III. Calculating the index of flatness. Suppose that the defining function $\langle H, E\rangle$ exhibits M as flat to order m at p. We now determine conditions on E_m that enable us to straighten M further. First we must deal with the exceptional cases $m = 0$ and $m = 1$. (The case $m = 0$ is further investigated to §IV.)

1. LEMMA. *Suppose that the defining function $\langle H, E\rangle$ does not exhibit M as flat to order 1 at p. Then the same is true for all defining functions and $F(M,p) = 0$.*

PROOF. Let $\langle h, e\rangle$ be another defining function. By Proposition II.4, and the fact that H and h vanish simultaneously (II.4.1), we obtain the formula

$$E_{00} = e_{00}(1 + 2\,\text{Re}(a_0))(1 + B_{00}), \tag{1.1}$$

where each term is expanded in terms of either variable, h or H. Equation (1.1) shows that E_{00} is a unit times e_{00}, so that these terms also vanish simultaneously. This shows that the ideal $(H, \overline{H}, E_{00})$ is also an invariant of M, and implies in particular the conclusion of the lemma.

2. PROPOSITION. *Suppose that $\langle H, E\rangle$ exhibits M as flat to order* 1 *at* p. *Then* $F(M,p)$ *is strictly larger than one if and only if*

$$E_{10}(1+a_0)-\overline{a}_0 \text{ is real valued, and lies in } |\mathcal{M}_p|^2.$$

Here $\overline{a}_0 = \pi_{01}(E_{10})$.

PROOF. By an application of Proposition II.4, the condition that $\langle h, e\rangle$ exhibit M as flat to order greater than one can be expressed as

$$E_1 = j_1(2\,\mathrm{Re}(H)B + (1+B)|1+a|^{-2}2\,\mathrm{Re}(H\overline{\alpha})), \tag{2.1}$$

where a and B define the usual unit, and (2.1) is expanded in terms of z_H. Applying (2.1) to E_{10} gives

$$E_{10} = B_{00} + (1+B_{00})\overline{a}_0(1+a_0)^{-1}, \tag{2.2}$$

because $\alpha = a + a^2$ and $E_{10} = \overline{E}_{01}$. Manipulation of (2.2) yields

$$E_{10}(1+a_0) - \overline{a}_0 = B_{00}(1+2\,\mathrm{Re}(a_0)). \tag{2.3}$$

Now, by Lemmas II.2 and II.3, $\pi_{10}(E_{10}) = 0$, and B_{00} is real and in $|\mathcal{M}_p|^2$. Hence (2.3) implies that the desired expression is real, lies in $|\mathcal{M}_p|^2$, and that $\overline{a}_0 = \pi_{01}(E_{10})$. This is the desired necessity result.

Conversely, if the left side of (2.3) satisfies these conditions, where $\overline{a}_0$ equals $\pi_{01}(E_{10})$, we define $B = B_{00}$ by (2.3), and $a = a_0$. The same application of (2.3) shows that

$$\langle H, E\rangle/(1+2\,\mathrm{Re}(a))(1+B) = \langle h, e\rangle, \tag{2.4}$$

where $e_{00} = e_{01} = e_{10} = 0$. This proves the proposition.

3. EXAMPLE. Consider $r = 2\,\mathrm{Re}(z_2) + 2\,\mathrm{Re}(z_2(\overline{z}_1 + f(z_1,\overline{z}_1)))$, where p is the origin and $f \in |\mathcal{M}_p|^2$. Then $F(M,p)$ is larger than one if and only if

$$(1+z_1)f(z_1,\overline{z}_1) \text{ is real.} \tag{3.1}$$

Note that this depends on the full Taylor series of f. We also see that it is possible for the index of flatness to be one.

Now we assume that $m \geq 2$, and that $\langle H, E\rangle$ exhibits M as flat to order m. We find the necessary and sufficient conditions for the existence of a new defining function $\langle h, e\rangle$ that exhibits M as flat to order $m+1$. By Proposition II.4, we have the formula, with $\alpha = a + a^2$,

$$E_k = 2\,\mathrm{Re}(H)B_{k-1} + ((1+B)|1+a|^{-2}2\,\mathrm{Re}(H\overline{\alpha}))_k, \qquad k \leq m. \tag{4.1}$$

Of course, (4.1) also vanishes for $k < m$.

5. LEMMA. *Suppose $m \geq 2$, and that* (4.1) *holds. Then*

$$B_{00} = 0, \tag{5.1}$$

$$\alpha_k = (-1)^k\overline{\alpha}_k \text{ is a constant for } k \leq m-2. \tag{5.2}$$

PROOF. We apply (4.1) when k equals one to get

$$0 = B_{00}(H+\overline{H}) + (1+B_{00})|1+a_0|^{-2}2\,\mathrm{Re}(\alpha_o\overline{H}). \tag{5.3}$$

This equation implies that $2\,\mathrm{Re}(\alpha_0\overline{H})$ is divisible by $H+\overline{H}$. By (II.9.1), $\alpha_0-\overline{\alpha}_0$ vanishes. Since α is holomorphic, it is a constant. Since $a_0\in\mathcal{M}_p$ and $\alpha_0=a_0+a_0^2$, α_0 vanishes at p, hence identically. Putting this in (5.2) tells us that B_{00} vanishes also. To prove (5.2), we assume inductively that the result holds, as we have just proved it when k vanishes. Assume (5.2) holds for $0<i<k\le m-2$. Using (4.1),

$$0=E_{i+2}=2\,\mathrm{Re}(H)B_{i+1}+\sum_{0}^{i+1}((1+B)|1+a|^{-2})_{1+i-s}2\,\mathrm{Re}(\alpha_s H^s\overline{H}). \tag{5.4}$$

By (5.1) and the case of (5.2) when $k=0$, $((1+B)|1+a|^{-2})_0=1$. Equation (5.4) then gives

(5.5)

$$2\,\mathrm{Re}(\alpha_{i+1}H^{i+1}\overline{H})=-2\,\mathrm{Re}(H)B_{i+1}-\sum_{s=0}^{i}((1+B)|1+a|^{-2})_{i+1-s}2\,\mathrm{Re}(\alpha_s H^s\overline{H}).$$

By the induction hypothesis and (II.9.1), each term on the right of (5.5) is divisible by $2\,\mathrm{Re}(H)$. Hence, so is the left side, and therefore, again using (II.9.1), we obtain the formula (5.2) when i equals $i+1$. Again, since α_{i+1} is holomorphic, it is a constant. This completes the induction.

6. THEOREM. *Suppose that M is a real analytic hypersurface of $\mathbf{C}^n$ and that p lies in M. Suppose that the defining function $\langle H,E\rangle$ exhibits M as flat to order m at p, and that $m\ge 2$. Then, for $F(M,p)$, the index of flatness at p, to equal m, it is necessary and sufficient that* $\mathrm{Alt}(E_m)$ *is not pluriharmonic.*

PROOF. First we suppose that $F(M,p)$ is strictly larger than m. Then, by (4.1), we have

$$E_m=2\,\mathrm{Re}(H)B_{m-1}+((1+B)|1+a|^{-2}2\,\mathrm{Re}(H\overline{\alpha}))_m. \tag{6.1}$$

By Lemma 5, we see that each term on the right of (6.1) is divisible by $H+\overline{H}$ except the term T, where

$$T=(2\,\mathrm{Re}(\alpha_{m-1}H^{m-1}\overline{H}))(1+B_{00})|1+a_0|^{-2}. \tag{6.2}$$

Taking Alt of both sides of (6.1) then gives

$$\mathrm{Alt}(E_m)=\mathrm{Alt}(T)=\left(\sqrt{-1}\right)^m(-\overline{\alpha}_{m-1}+(-1)^{m-1}\alpha_{m-1}), \tag{6.3}$$

because of Lemma II.9 and Lemma 5. It follows from (6.3) that $\mathrm{Alt}(E_m)$ is $\pm\mathrm{Re}(\alpha_{m-1})$ or $\pm\mathrm{Im}(\alpha_{m-1})$ depending on the value of m mod 4. In all cases, however, $\mathrm{Alt}(E_m)$ is pluriharmonic, because α_{m-1} is holomorphic.

Conversely, we suppose that $\mathrm{Alt}(E_m)$ is pluriharmonic, and define α_{m-1} to be the holomorphic function that satisfies (6.3). Define a by the formula $a+a^2=\alpha_{m-1}H^{m-1}$, with $a(p)=0$. Note that $\alpha_{m-1}=a_{m-1}$. Also, let $h=H(1+a)^{-1}$. Consider $S=E_m-2\,\mathrm{Re}(\alpha_{m-1}H^{m-1}\overline{H})$. By Lemma II.9, S is divisible by $2\,\mathrm{Re}(H)$. Call the quotient B_{m-1}. Since S is real valued, so is B_{m-1}. To show that B_{m-1} lies in $|\mathcal{M}_p|^2$, it is enough to show that $\pi_{10}(B_{(m-1)0})=0$. If this term did not vanish, however, then S would include a term of the form $\pi_{10}(B_{(m-1)0})H^m$, which contradicts the fact that $\pi_{10}(E_{m0})=0$. (This last fact holds because $E_{m0}\in|\mathcal{M}_p|^2$.) Now the defining function $\langle H,E\rangle/(1+2\,\mathrm{Re}(a))(1+B)$ satisfies the condition $j_m(e,z_h)=0$ by construction and equation (II.4.2).

7. EXAMPLE. Suppose that n equals one. Then it follows from the Cauchy-Kowaleski theorem that the index of flatness is infinite. To see that our theorem is consistent with this result, note that the condition in case m equals one is automatic, because E_{10} must vanish as it is a constant. Now note that $\mathrm{Alt}(E_m)$ is also a constant, and real, so that it is pluriharmonic. Thus we can straighten M to arbitrarily high order. By a careful analysis of the size of the coefficients, we see that the resulting infinite product defining h must converge. In other words, Theorem 6 gives an algebraic procedure for straightening up to certain obstructions, which necessarily vanish in one dimension. Thus there is a sequence of coordinate changes $H \mapsto H/(1+a) = h$ that straighten to arbitrarily high order. We do not produce the analysis here that guarantees convergence.

8. EXAMPLE. Put $n = 2$, and $r = 2\,\mathrm{Re}(z_2) + |z_1^p z_2|^{2m}$. Then the index of flatness at the origin is $2m$, no matter what p is, if $p \geq 1$. The reason is that $E_{mm} = |z_1|^{2mp}$ is not pluriharmonic, and since all other E_{ik} vanish, $\mathrm{Alt}(E_{2m})$ is not pluriharmonic. Note that M contains the complex line $z_2 = 0$, so that in the sense of order of contact, M is "flat" at the origin. We also note that equally trivial examples show that the index of flatness can be any odd integer as well.

Before turning to the case where E_{00} does not vanish, we briefly study the completely flat case. The following easy theorem is essentially a restatement of Proposition II.4, but little more can be said.

9. THEOREM. *Suppose that M is a real analytic hypersurface of $\mathbf{C}^n$ that contains p. Let $\langle H, E\rangle$ be a defining function. Then M is completely Levi flat at $p \Leftrightarrow$ there is a real B in $|\mathcal{M}_p|^2$ and an a in $\mathcal{M}_p$ so that*

$$E = B \cdot 2\,\mathrm{Re}(H) + (1+B)2\,\mathrm{Re}(\overline{a}H(1+a)^{-1}). \tag{9.1}$$

In particular, it is necessary that, for some $\alpha \in \mathcal{M}_p$,

$$E \equiv \mathit{unit} \cdot \mathrm{Im}(\alpha)2\,\mathrm{Im}(h) \quad \mathrm{mod}(H + \overline{H}). \tag{6.2}$$

Example 10 below shows that (9.2) is not sufficient. In fact, α and the unit in (9.2) determine a and B in (9.1).

PROOF. M is completely Levi flat at $p \Leftrightarrow M$ can be defined by $\langle h, 0\rangle$. Put $e = 0$ into (II.4.2) and (9.1) results. To show the necessity of (9.2), compute (9.1) $\mathrm{mod}(2\,\mathrm{Re}(H))$, and get

$$\begin{aligned} E &\equiv (1+B)|1+a|^{-2}(\alpha\overline{H} + \alpha\overline{H}) && \mathrm{mod}\, 2\,\mathrm{Re}(H) \\ &\equiv \mathrm{unit} \cdot (\overline{\alpha} - \alpha)(H - \overline{H})/2 && \mathrm{mod}\, 2\,\mathrm{Re}(H) \\ &\equiv \mathrm{unit} \cdot \mathrm{Im}(\alpha)2\,\mathrm{Im}(H) && \mathrm{mod}\, 2\,\mathrm{Re}(H). \end{aligned} \tag{9.3}$$

Given (9.3), $a + a^2 = \alpha$ and $a(p) = 0$ determine a. The unit must then be $(1+B)|1+a|^{-2}$, so the last remark follows.

10. EXAMPLE. $r = 2\,\mathrm{Re}(z_2) + 2\,\mathrm{Re}(\overline{z}_1 z_2(1+z_1)^{-1})$. Then $E_{10} - \pi_{01}(E_{10}) = \overline{z}_1(1+z_1)^{-1} - \overline{z}_1 = -|z_1|^2(1+z_1)^{-1}$ is not real. Hence, although (9.2) holds with $\alpha = z_1 + a_1^2$, M is not completely flat. In fact, $F(M,p)$ equals one by Proposition 2.

11. REMARK. Freeman [**10**] has an alternative list of obstructions to straightening. It would be interesting to determine the precise relationships.

IV. Tanaka regularity. The first obstruction to flatness is the E_{00} term. We now consider those hypersurfaces for which this term is nonzero. In fact, any hypersurface that does not contain a copy of C^{n-1} passing through p satisfies $E_{00} \neq 0$. Tanaka considered hypersurfaces for which $r = 2\operatorname{Re}(H) + E_{00}$, so we call such a defining equation T-regular. In the spirit of §III, we make the following

1. DEFINITION. Let M be a real analytic hypersurface of $\mathbf{C}^n$, and let p be a point in M. We say that the defining function $\langle H, E\rangle$ and local coordinate system z_H exhibit M as T-regular to order m at p if $j_{m-1}(E, z_H) = E_{00}$. The index of T-regularity $T(M,p)$ is the largest integer m for which there is a defining function and corresponding coordinates that exhibit M as T-regular to order m at p. If no such integer exists, we say $T(M,p)$ equals infinity.

2. REMARKS. There is no reason to consider the case $m = 0$, because, by Lemma III.1, if E_{00} does not vanish, then e_{00} does not either for any new defining function. Thus every hypersurface is automatically T-regular to order at least one. Also we emphasize that here it is indispensable to mention the dependence on the coordinate system.

In the spirit of §III we determine when $T(M,p)$ equals m, given that $j_m(E, z_H) = E_{00} + E_m$. As before, we consider the case $m = 1$ separately. First we consider the general lemma.

3. LEMMA. *Suppose that $\langle H, E\rangle$ exhibits M as T-regular to order m at p. Then $T(M,p) > m \Leftrightarrow$ there is an $a \in \mathcal{M}_p$ and there is a real $B \in |\mathcal{M}_p|^2$ for which*

$$(3.1)\quad E_{00} + E_m = 2\operatorname{Re}(H)B + 2\operatorname{Re}(H\overline{a})(1+B)|1+a|^{-2} + E_{00}((1+2\operatorname{Re}(a))(1+B)(1+2\operatorname{Re}(a_0))^{-1}(1+B_{00})^{-1}),$$

where we take j_m of the right side with respect to z_H. (We omit the subscript.)

PROOF. $T(M,p) > m \Leftrightarrow$ there exists $\langle h, e\rangle$ for which $j_{m+1}(e, z_h) = e_{00}$. Choosing the usual unit $(1+2\operatorname{Re}(a))(1+B)$ gives (3.1), if we also use (III.1.1).

4. PROPOSITION. $T(M,p) > 1 \Leftrightarrow$ *there is a real C in $|\mathcal{M}_p|^2$ and a real w in $\mathcal{M}_p \oplus \overline{\mathcal{M}}_p$ so that, with $\overline{a}_0 = \pi_{01}(E_{10})$,*

$$(4.1)\qquad (1+a_0)E_{10} - a_0 = C + (1+a_0)wE_{00}.$$

In particular, it is necessary that the left side of (4.1) *be congruent to a real valued element of* $|\mathcal{M}_p|^2 \bmod E_{00}$.

PROOF. From (3.1) we obtain the necessary and sufficient condition that

$$(4.2)\qquad E_{10} = B_{00} + \overline{a}_0(1+B_{00})(1+a_0)^{-1} + E_{00}(a_1(1+B_{00})^{-1} + B_{10}(1+2\operatorname{Re}(a_0))^{-1}).$$

Now (4.2) forces $\overline{a}_0$ to be $\pi_{01}(E_{10})$. Solving for the left side of (4.1), we see that $C = (1+2\operatorname{Re}(a_0))B_{00}$. Also the coefficient of E_{00} in (4.2) is easily seen to be an arbitrary real valued nonunit w of $\mathcal{A}_p$, because $a_1 = \pi_{10}(w)$ and $\pi_{01}(B_{10}) = \pi_{01}(w)$. These relationships enable us to adjust a_1 and B_{10} to accomodate any w.

5. EXAMPLE. Put $r = 2\operatorname{Re}(z_2) + |z_1|^4 + |z_1|^2 2\operatorname{Re}(z_1 z_2)$; $p =$ origin. Thus $E_{00} = |z_1|^4$, $E_{10} = |z_1|^2 z_1$, $\pi_{01}(E_{10}) = 0$. Thus the left side of (4.1) is $|z_1|^2 z_1$, whose residue class $\operatorname{mod}(E_{00})$ is not real. Thus $T(M,p) = 1$.

Now we suppose that $\langle H, E\rangle$ exhibits M as T-regular to order 2 or more; we obtain many simplifications in (3.1). First we obtain the following stronger version of Lemma III.5.

6. LEMMA. *Suppose that $m \geq 2$, and that the hypotheses of Lemma 3 hold. Then a and B in (3.1) must satisfy, for $0 \leq k \leq m-2$,*

$$0 = a_k, \tag{6.1}$$

$$0 = B_{k0} + E_{00}(a_{k+1} + (1+B_{00})^{-1}B_{(k+1)0}), \tag{6.2}$$

$$B_{k0} \text{ is real,} \tag{6.3}$$

$$\pi_{01}(B_{(k+1)0}) = \overline{a}_{k+1}. \tag{6.4}$$

PROOF. First we note that (6.2) and (6.3) together imply (6.4). To see this, (6.2) implies that $a_{k+1} + (1+B_{00})^{-1}B_{(k+1)0}$ is real. This is possible only if $\pi_{01}(B_{(k+1)0}) = \overline{a}_{k+1}$, because of Lemma II.2, and the fact that $\pi_{01}(B_{00})$ vanishes. Thus we only need to check the first three properties, which we do by induction. When $k = 0$, we set $E_1 = 0$. Using (3.1) we obtain

$$\begin{aligned} 0 = B_{00} &+ \overline{\alpha}_0(1+B_{00})|1+a_0|^{-2} \\ &+ E_{00}(a_1(1+2\operatorname{Re}(a_0))^{-1} + B_{10}(1+B_{00})^{-1}). \end{aligned} \tag{6.5}$$

Computing (6.5) mod$|\mathcal{M}_p|^2$, we get $\alpha_0 = 0$, hence $a_0 = 0$. Putting this back in (6.5), we get

$$0 = B_{00} + E_{00}(a_1 + B_{10}(1+B_{00})^{-1}). \tag{6.6}$$

Since B_{00} is automatically real, we now have the three properties when $k = 0$. Now suppose that the result holds up to k. If k equals $m-2$, we are done. Otherwise E_{k+2} vanishes. Using the induction hypothesis, (3.1) becomes

$$\begin{aligned} 0 = B_{k+1}2\operatorname{Re}(H) &+ 2\operatorname{Re}(\alpha_{k+1}H^{k+1}\overline{H})(1+B_{00}) \\ &+ E_{00}(2\operatorname{Re}(a_{k+2}H^{k+2}) + 2\operatorname{Re}(a_{k+1}H^{k+1})B_1 + B_{k+2})(1+B_{00})^{-1}. \end{aligned} \tag{6.7}$$

Equation (6.7) gives

$$0 = B_{(k+1)0} + E_{00}(a_{k+2} + a_{k+1}B_{10} + B_{(k+2)0})(1+B_{00})^{-1}. \tag{6.8}$$

Now we see from (6.8) that $B_{(k+1)0}$ lies in $|\mathcal{M}_p|^2$, hence $\pi_{01}(B_{(k+1)0})$ vanishes; by (6.4) and the induction hypothesis, $a_{k+1} = 0$. Thus (6.8) gives (6.2) with k replaced by $k+1$. Now if B_{k0} is real, (6.2) holds, and $a_{k+1} = 0$, we must have $B_{(k+1)0}$ also real. This completes the induction.

This lemma gives easily an apparently stronger

7. COROLLARY. *Under the hypotheses of Lemma 6, we have*

$$\text{for } 1 \leq k \leq m-2, \quad 0 = B_{k-1}2\operatorname{Re}(H) + E_{00}(B_k(1+B_{00})^{-1}), \tag{7.1}$$

$$0 = B_{m-2}2\operatorname{Re}(H) + E_{00}(2\operatorname{Re}(a_{m-1}H^{m-1}) + B_{m-1}(1+B_{00})^{-1}), \tag{7.2}$$

$$\begin{aligned} E_m = B_{m-1}2\operatorname{Re}(H) &+ 2\operatorname{Re}(\alpha_{m-1}H^{m-1}\overline{H})(1+B_{00}) \\ &+ E_{00}(2\operatorname{Re}(a_mH^m) + 2\operatorname{Re}(a_{m-1}H^{m-1})B_1 + B_m). \end{aligned} \tag{7.3}$$

PROOF. Using (3.1), and the fact that $j_{m-2}(a, z_H)$ vanishes, (7.1) and (7.2) are simply the statement that E_k vanishes. (Equation (7.2) is slightly different because we do not know that a_{m-1} vanishes.) Equation (7.3) is also (3.1) applied to E_m, which does not vanish in general.

8. THEOREM. *Let M be a real analytic hypersurface of $\mathbf{C}^n$ that contains the point p. Suppose that $\langle H, E\rangle$ is a defining equation for M that exhibits M as T-regular to order m at p, with $m \geq 2$. Then, for $T(M,p)$ to be larger than m, it is necessary and sufficient that E_m have the following form:*

$$E_m \equiv t(H+\overline{H})^m + 2\operatorname{Re}(a_{m-1}H^{m-1}\overline{H}) \quad \operatorname{mod}(E_{00}). \tag{8.1}$$

In (8.1), $a_{m-1} \in \mathcal{O}_p$, $t \in |\mathcal{M}_p|^2$ is real, and each is independent of H.

PROOF. *Necessity.* First we study the effect of the equations (7.1), (7.2) and (7.3). Since $j_{m-2}(a)$ vanishes, $a_{m-1} = \alpha_{m-1}$. Next use (7.1) to express each B_k, for $1 \leq k \leq m-2$, in terms of B_{00}. Then eliminate B_{m-1} using (7.2). The result is, after some manipulation,

(8.2)
$$\begin{aligned} E_m = (-1)^{m-1}(1+B_{00})^{m-1}B_{00}(E_{00}^{m-1})^{-1}(H+\overline{H})^m + 2\operatorname{Re}(a_{m-1}H^{m-1}\overline{H}) \\ + E_{00}(2\operatorname{Re}(a_m H^m) + B_m(1+B_{00})^{-1}) - B_{00}(2\operatorname{Re}(a_{m-1}H^m)). \end{aligned}$$

From (8.2), it follows that $B_{00} \in (E_{00}^{m-1})$. Define real valued, real analytic functions w and t by

$$B_{00} = wE_{00}^{m-1}, \tag{8.3}$$

$$t = (-1)^{m-1}(1+wE_{00}^{m-1})^{m-1}w. \tag{8.4}$$

Now, computing (8.2) $\operatorname{mod}|\mathcal{M}_p|^2$ shows that w and t are in $|\mathcal{M}_p|^2$. Also, the last term is of course in (E_{00}), so substitution into (8.2) gives

$$E_m \equiv t(H+\overline{H})^m + 2\operatorname{Re}(a_{m-1}H^{m-1}\overline{H}) + E_{00}(\text{stuff}). \tag{8.5}$$

Of course, this proves the necessity of (8.1).

Sufficiency. We must show that, for any real choice of "stuff" in (8.5), and any choice of t and a_{m-1}, we can choose a_m and B_m to make (8.2) valid. Of course we must have a_m holomorphic, B_m real and in $|\mathcal{M}_p|^2$. Once we do this, we substitute the equations of Corollary 7, and using Lemma 6, see that (3.1) holds. This implies the result. First we note that equation (8.4) exhibits t as a unit in w times w. It follows from the implicit function theorem that there is a unique real analytic germ w that satisfies (8.4) and lies in $|\mathcal{M}_p|^2$. We have used the fact that t lies in $|\mathcal{M}_p|^2$, which follows from computing (8.1) $\operatorname{mod}|\mathcal{M}_p|^2$. Thus (8.4) determines w, and define B_{00} by (8.3). Furthermore, we define $a_m H^m$ by $\pi_{10}(\text{stuff})$. Then we have

$$\text{stuff} - 2\operatorname{Re}(a_m H^m) \in |\mathcal{M}_p|^2 \text{ and is real.} \tag{8.6}$$

We have used the fact that $w \in |\mathcal{M}_p|^2$ in studying the last term in (8.2). Now the expression (8.6) shows that we can define B_m to make (8.2) hold, and still have B_m real and in $|\mathcal{M}_p|^2$. Thus we can accomodate any coefficient of E_{00}, and the result follows.

The condition of Theorem 8 is quite strong. Note that the first term in (8.1) lies in the ideal $(H+\overline{H})^m$, rather than simply in $(H+\overline{H})$ as happens in the flat case. This makes it very easy to check. We have many corollaries; the first is a uniqueness result that will be needed when we try to put E_{00} into a normal form.

9. COROLLARY. *Suppose that* $\langle H, E\rangle$ *and* $\langle h, e\rangle$ *both exhibit* M *as* T*-regular at* p. *Then* $H = h$ *and* $E = e$.

PROOF. Since $\langle H, E\rangle$ exhibits M as T-regular at p, we have, for each m, Lemma 6, Corollary 7, and (8.2). By (6.1), a_k vanishes for all k, and, since a is holomorphic, it vanishes also. Now (8.2) implies that B_{00} lies in $(E_{00})^k$ for all k larger than or equal to one. Thus, by Krull's lemma, B_{00} vanishes identically. Using equation (7.1), we write $0 = B_{k-1}(H + \overline{H}) + E_{00}(B_k)$, which by induction implies that $B_k = 0$ for all k, hence $B = 0$. Therefore the unit under consideration is the constant "one", so the result follows from (II.3.1).

This corollary means that once h is determined, the function e is then determined uniquely. This function can be reexpressed by any coordinate change in the $n - 1$ variables complementary to h. Thus we have not determined a system of coordinates.

10. COROLLARY. *Suppose that the hypotheses of Theorem* 8 *hold. A necessary condition for* $T(M, p)$ *to be larger than* m *is that*

$$\text{(10.1)} \qquad \operatorname{Alt}(E_m) \equiv \textit{a pluriharmonic function} \quad \operatorname{mod}(E_{00}).$$

PROOF. Take Alt of both sides of (8.1); the result follows as in (III.6.3).

11. COROLLARY. *Suppose that* $\operatorname{Alt}(E_m)$ *lies in* (E_{00}). *Then a necessary condition for* $T(M, p)$ *to be larger than* m *is that* E_m *lies in the ideal* $((H + \overline{H})^m, E_{00})$.

PROOF. This follows immediately from (8.1).

We also have the analog of Theorem III.9. Again (12.1) is quite easy.

12. THEOREM. *Suppose* M *is a real analytic hypersurface of* $\mathbf{C}^n$ *that contains* p. *Let* $\langle H, E\rangle$ *be a defining function for* M. *For* M *to be* T*-regular at* p, *it is necessary and sufficient that, with* $\alpha = a + a^2$

$$\text{(12.1)} \qquad \begin{aligned} E = {} & E_{00}(1 + 2\operatorname{Re}(a))(1 + B)(1 + 2\operatorname{Re}(a_0))^{-1}(1 + B_{00})^{-1} \\ & + B - 2\operatorname{Re}(H) + (1 + B)|1 + a|^2 2\operatorname{Re}(H\overline{\alpha}) \end{aligned}$$

for some $a \in \mathcal{M}_p$ *and* $B \in |\mathcal{M}_p|^2$. *In particular it is necessary, but not sufficient, that*

$$\text{(12.2)} \qquad \begin{aligned} E &\equiv \textit{unit} \cdot 2\operatorname{Re}(H\overline{\alpha}) \quad \operatorname{mod}(E_{00}, H + \overline{H}) \\ &\equiv \textit{unit} \cdot \operatorname{Im}(\alpha) 2\operatorname{Im}(H) \quad \operatorname{mod}(E_{00}, H + \overline{H}). \end{aligned}$$

PROOF. Again the result is essentially only a restatement of (II.4.2). The details are essentially the same as in III.9 and hence are omitted.

13. COROLLARY. *Suppose that* M *is a real analytic hypersurface of* $\mathbf{C}^n$ *that contains* p. *Let* $\langle H, E\rangle$ *be a defining function. In order that* M *be* T*-regular at* p, *it is necessary that there exists a pluriharmonic function* v *so that* E *lies in the ideal* $(E_{00}, v, H + \overline{H})$. (*Recall that the ideals* (E_{00}) *and* (H) *are invariants of* M *at* p.)

PROOF. This follows immediately from (12.2), with $v = \operatorname{Im}(\alpha)$.

V. Examples. Now we combine the results of §§III and IV with the methods of [**9**]. We determine whether a given defining function defines the same hypersurface as do certain prescribed special cases. We omit the proofs of the next two lemmas, which were originally used in the study of order of contact. The proofs appear in [**9, 8**].

1. LEMMA. *Suppose E_{00} is a real valued element of $|\mathcal{M}_p|^2$. Then there is a Hilbert space $\mathcal{H}$ and holomorphic $\mathcal{H}$ valued functions F, G so that $F(p) = G(p) = 0$, and*

$$E_{00} = \|F\|^2 - \|G\|^2. \tag{1.1}$$

(*This expression is not unique.*).

2. LEMMA. *Suppose that α, β are $\mathcal{H}$-valued holomorphic functions, with $\alpha(p) = \beta(p) = 0$, $\|\alpha\|^2 = \|\beta\|^2$. Then there is a unitary operator U (with constant entries) on $\mathcal{H}$ for which $\alpha = U\beta$.*

3. DEFINITION. Let $w = (w_1, \ldots, w_N)$ denote a finite set of elements of $\mathcal{M}_p$, none of which vanishes. N is called minimal if there does not exist K, with $K < N$, so that $\|w\|^2 = \|z\|^2$, and $z = (z_1, \ldots, z_K)$.

4. LEMMA. *N is minimal $\Leftrightarrow$ there exists no unitary matrix U on $\mathbf{C}^n$, so that $\sum_{j=1}^N U_{ij} w_j$ vanishes for any i.*

PROOF. By Lemma 2, $\|w\|^2 = \|z\|^2 \Leftrightarrow z = Uw$ for some unitary U. This implies that $\sum U_{ij} w_j = 0$ for $i > K$, $z_i = \sum U_{ij} w_j$, $i \le K$. Hence, if such a unitary exists, these equations show that N is not minimal, and if N is not minimal, they force the existence of such a matrix.

We now consider some "partial" normal forms. Suppose that w is given as in Definition 3, with N minimal. We ask whether our given defining function $\langle H, E\rangle$ defines the same hypersurface as does equation (5.1) below. We assume p is the origin, and $w(p) = 0$,

$$2\,\mathrm{Re}(z_n) + \sum_{j=1}^N |w_j(z')|^2, \quad \text{where } z' = (z_1, z_2, \ldots, z_{n-1}). \tag{5.1}$$

Step 1. Suppose $E_{00} = 0$. Then (5.1) is possible only if $w = 0$. If in fact this is so, we use the methods of §III to determine the index of flatness. If $F(M,p) < \infty$, (5.1) is impossible. Otherwise, (5.1) holds with $w = 0$.

Step 2. Suppose $E_{00} \neq 0$. Let m_0 be the smallest positive integer for which $j_m(E, z_H) \neq E_{00}$. If no such m_0 exists, go to Step 3. Otherwise, we apply the necessary and sufficient conditions for T-regularization given in §IV. If $T(M,p) < \infty$, (5.1) is impossible. Otherwise,

Step 3. Suppose we now have $2\,\mathrm{Re}(h) + e_{00}$. According to Corollary IV.9, we must have $e_{00} = \|w\|^2$. Write $e_{00} = \|f\|^2 - \|g\|^2$ according to (1.1). According to Lemma 2, we have

$$\begin{pmatrix} 0 \\ F \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} w \\ G \end{pmatrix}, \quad \text{where the matrix is unitary.} \tag{5.2}$$

Equation (5.2) implies that $F = Cw + DG$, and upon taking conjugate transposes, that $G = D^* F$. This gives, again using the unitarity,

$$(1 - DD^*)(F) = CC^* F = Cw. \tag{5.3}$$

If C vanishes in (5.3), we have D is unitary and $\|F\|^2 = \|G\|^2$, so e_{00} vanishes, which is a contradiction to (III.1.1). (We are assuming $E_{00} \neq 0$.) Otherwise $C \neq 0$, and $CC^* < I$. Conversely we can always extend such a C to be part of a unitary matrix as in (5.2). Therefore we have $e_{00} = \|w\|^2 \Leftrightarrow$ there is such a C with $CC^*F = Cw$.

We now seek some sort of normal form for $\|w\|^2$, i.e., some choice of local coordinates for which this term is as simple as possible in an appropriate sense. We hope to accomplish this in a future paper, as well as the understanding of the general case. We conclude with an example.

6. EXAMPLE. Put $r(z) = 2\operatorname{Re}(z_n) + |z_n|^2 + \|f(z')\|^2$. Then the considerations of §III make us set

$$z_n = w_n\left(1 - \tfrac{1}{2}w_n\right), \tag{6.1}$$

$$u = \left|1 - \tfrac{1}{2}w_n\right|^{-2}, \tag{6.2}$$

which give

$$2\operatorname{Re}(w_n) + \left\|\left(1 - \tfrac{1}{2}w_n\right)^{-1} f(w')\right\|^2. \tag{6.3}$$

Now if we could write the vector valued function $(1 - \frac{1}{2}w_n)f(w')$ in terms of new coordinates, say, $z_1, \dots, z_{n-1}$, then we could have a T-regular surface. One case where this occurs is where $f(w) = (w_1^{p_1}, w_2^{p_2}, \dots, w_{n-1}^{p_{n-1}})$. For then, we could define

$$z_j = w_j\left(1 - \tfrac{1}{2}w_n\right)^{-1/p_j}. \tag{6.4}$$

Thus, for example, the normal form for $\sum^{n-1}|z_j|^{2p_j} + |\tilde{z}_n|^2 < 1$ at the point $(0, 0, \dots, 1)$, where $\tilde{z}_n = z_n + 1$, becomes

$$2\operatorname{Re}(z_n) + \sum^{n-1}|z_j|^{2p_j} \text{ at the origin.} \tag{6.5}$$

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